

Massless metric preheating

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Can super-Hubble metric perturbations be amplified exponentially during preheating? Yes. An analytical existence proof is provided by exploiting the conformal properties of massless inflationary models. The traditional conserved quantity ζ is non-conserved in many regions of parameter space. We include backreaction through the homogeneous parts of the inflaton and preheating fields and discuss the rôle of initial conditions on the post-preheating power-spectrum. Maximum field variances are strongly underestimated if metric perturbations are ignored. We illustrate this in the case of strong self-interaction of the decay products. Without metric perturbations, preheating in this case is very inefficient. However, metric perturbations increase the maximum field variances and give alternative channels for the resonance to proceed. This implies that metric perturbations can have a large impact on calculations of relic abundances of particles produced during preheating.

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I. INTRODUCTION

Reheating is a crucial element of any inflationary cosmology. It endows the nascent universe with the high temperatures and large entropy needed for its subsequent evolution. Reheating changes the equation of state from the near de Sitter $p \simeq -\rho$, to the radiation-dominated form $p = \frac{1}{3}\rho$. This leads to a large amplification of the power spectrum of super-Hubble metric perturbations resulting in a final value $P_k \propto \dot{\phi}^{-1}|_{k=aH}$, where ϕ is the slowly-rolling inflaton field [1,2]. This we call the ‘old’ theory of cosmological reheating. It is a theory where coherence of the inflaton, and the fact that the Hubble scale, H^{-1} , at the end of inflation is dwarfed by the vastly larger particle horizon d_H , play no rôle.

In contrast, the super-Hubble coherence [3] of the inflaton condensate forms the foundation of preheating [4]. It is an intrinsically non-equilibrium era [5] in which stimulated, coherent processes can totally dominate single-body, perturbative decays. The resulting phase of violent particle production leaves as its legacy a non-thermal spectrum of particles. The subsequent backreaction and rescattering [6] of these fields leads to turbulence and thermalisation [7].

While many different models have been proposed, the basic mechanism of parametric resonant amplification of quantum fluctuations is very robust and bears many similarities to standard quantum field theory in strong external fields [8,9]. Given this robustness, a natural question is whether these time-dependent mass effects have implications for the standard predictions of inflationary cosmology *. Primary among these predictions is the

spectrum of metric perturbations, which will provide precision tests of the inflationary paradigm. It is therefore of great interest to understand these issues in detail.

That parametric resonance due to oscillating scalar fields might have an interesting impact on metric perturbations was first noted by Kodama and Sasaki [2]. The first works addressing this issue within the context of preheating however claimed that there would be little effect on metric perturbation evolution on super-Hubble scales, and in particular that ζ (see eq. 10), would remain conserved [10,11]. These conclusions stem from considering small resonance parameters $q \leq O(1)$ such as occurs in the single field $\lambda\phi^4$ model [4]. While these models can yield a rich spectrum of sub-Hubble physics [12] they cannot be considered generic models of preheating, which *usually* requires $q \gg 1$ in order to be efficient [6].

Detailed simulations [14] have presented strong evidence in support of the intuitive arguments [†] for the existence of robust resonances on super-Hubble (*a.k.a.* “super-horizon”) scales [13]. Such amplification does not violate causality [13,14]. Nevertheless, an *analytical* existence proof of the possibility of exponential growth of super-Hubble metric perturbations remained outstanding. This *Letter* aims to provide this missing link. We will show that preheating can in principle cause significant departures from the old theory of metric perturbation evolution through reheating.

The failure of the old theory is most glaringly seen

*It is important to distinguish between metric amplification due (1) purely to resonances which stem from reversible

changes to effective masses and (2) to thermalisation which involves irreversible change in the average equation of state and large entropy production.

[†]Such as the fact that gravity has *negative* specific heat: inhomogeneity is the natural end-point of gravitational evolution.

in the violent, essentially exponential[‡], growth of the Bardeen potential ζ . In the old theory, ζ is exactly conserved in the adiabatic, long-wavelength limit. This allowed one to match the scalar metric perturbation Φ across the desert separating the Grand Unified scale and photon decoupling where temperature anisotropies in the CMB were formed on large angles with $\Delta T/T \simeq \frac{1}{3}\Phi$ ($\Omega = 1, \Lambda = 0$).

In section (II) we set out the general class of models we consider together with the general, conformally rescaled multi-field equations. Section (III) generalises the work of [17,18] to include metric perturbations. This provides the basic analytical results of our paper. Section (IV) discusses the appropriate initial conditions for the χ field at the start of preheating while section (V) discusses the χ power spectrum at the start of preheating, one of the key factors in evaluating the impact of preheating on the CMB. Finally section (VI) discusses the effects of back-reaction and self-interaction among the decay products. We use natural units where $G = 1$.

II. THE GENERAL CLASS

Our general potential for the class of massless scalar fields φ_i , interacting with couplings g_{ijkl} , will be:

$$V(\varphi) = \sum_{i,j,l,m}^N g_{ijlm} \varphi_i \varphi_j \varphi_l \varphi_m. \quad (1)$$

This includes many interesting sub-classes such as the archetypal $\lambda\phi^4 + g^2\phi^2\chi^2$ model considered in section (III). We choose this form since it is the most general one consistent with conformal invariance[§]. The fluctuations in these fields generate scalar metric perturbations** which, in the longitudinal gauge, are encoded in the perturbed metric

$$ds^2 = a(\eta)^2 [(1 - 2\Phi) d\eta^2 - (1 + 2\Psi) dx^i dx_i], \quad (2)$$

where $d\eta = dt/a$ is the conformal time, $a(\eta)$ is the scale factor, and $\Phi = \Psi$ since the anisotropic stress of the system vanishes to linear order [14]. It proves very useful to re-scale all the fields by $a(\eta)$. We adopt the convention that $\tilde{F} \equiv aF$ for any field F . The spatially homogeneous parts of the fields satisfy the background Klein-Gordon equations:

[‡]By *essentially exponential* we mean exponential modulo sub-dominant power-law factors; i.e. $e^{\mu t}/t^\alpha$, $\alpha \leq 1$.

[§]We neglect the conformal coupling to the curvature since the associated resonance is weak for $\xi = 1/6$ [16].

**We neglect vector and tensor perturbations [20] and consider a flat FLRW background. See [18,19] for aspects of the non-flat case.

$$\tilde{\varphi}_i'' - \frac{a''}{a} \tilde{\varphi}_i + \mathcal{V}_{\tilde{\varphi}_i} = 0, \quad (3)$$

where $\prime \equiv d/d\eta$, $\mathcal{V}_{\tilde{\varphi}_i} \equiv \partial\mathcal{V}/\partial\tilde{\varphi}_i$ and $\mathcal{V}(\tilde{\varphi}) \equiv a^4 V(\varphi)$ is the conformal potential, given by

$$\mathcal{V}(\tilde{\varphi}) = \Sigma_{i,j,l,m} g_{ijlm} \tilde{\varphi}_i \tilde{\varphi}_j \tilde{\varphi}_l \tilde{\varphi}_m. \quad (4)$$

The perturbed, multi-field, Einstein equations are:

$$\delta\tilde{\varphi}_k^{i''} + k^2\delta\tilde{\varphi}_k^i + \sum_j \mathcal{V}_{\tilde{\varphi}_i\tilde{\varphi}_j} \delta\tilde{\varphi}_k^j = \frac{4}{a} \left[\tilde{\Phi}'_k \tilde{\varphi}^{i'} - \frac{1}{2} \mathcal{V}_{\tilde{\varphi}_i} \tilde{\Phi}_k \right] - \frac{4a'}{a^2} \left[\frac{\tilde{\Phi}\tilde{\varphi}}{a} \right]', \quad (5)$$

$$\tilde{\Phi}'_k = \frac{4\pi}{a} \sum_i (\tilde{\varphi}^{i'} - \mathcal{H}\tilde{\varphi}^i) \delta\tilde{\varphi}_k^i. \quad (6)$$

We now reformulate this system in terms of the rescaled Sasaki-Mukhanov variables:

$$\tilde{Q}_k^i \equiv \delta\tilde{\varphi}_k^i + \frac{\tilde{\varphi}^{i'}}{a'} \tilde{\Phi}_k - \frac{\tilde{\varphi}^i}{a} \tilde{\Phi}_k, \quad (7)$$

which satisfy the equations:

$$\tilde{Q}_k^{i''} + \left[k^2 - \frac{a''}{a} \right] \tilde{Q}_k^i + \sum_j \mathcal{V}_{\tilde{\varphi}_i\tilde{\varphi}_j} \tilde{Q}_k^j = \frac{8\pi}{a} \sum_j \tilde{M}^{ij} \tilde{Q}_k^j.$$

The constraint equation (6) can be recast as:

$$\tilde{\Phi}'_k + \left[2\mathcal{H} - \frac{a''}{a'} \right] \tilde{\Phi}_k = \frac{4\pi}{a} \sum_i (\tilde{\varphi}^{i'} - \mathcal{H}\tilde{\varphi}^i) \tilde{Q}_k^i, \quad (8)$$

where $\mathcal{H} = a'/a$. Note that when $a'' = 0$, the LHS of these equations are expansion-invariant (EI) while the non-EI coefficients on the RHS are:

$$\tilde{M}^{ij} \equiv \left[\frac{\tilde{\varphi}^{i'} \tilde{\varphi}^{j'}}{a'} \right]' + \frac{1}{a} [\tilde{\varphi}^{i'} \tilde{\varphi}^{j'} - (\tilde{\varphi}^i \tilde{\varphi}^j)'] + \frac{1}{a^2} [a' \tilde{\varphi}^i \tilde{\varphi}^j]' - \frac{a'^2}{a^3} [\tilde{\varphi}^i \tilde{\varphi}^j]. \quad (9)$$

Finally, the traditional “conserved” quantity ζ , rescaled by a , is given by:

$$\tilde{\zeta}_k = \tilde{\Phi}_k + \left[2 - \frac{a''}{a'} \right]^{-1} \mathcal{H}^{-1} \tilde{\Phi}'_k. \quad (10)$$

Equations (7-10) will provide the foundation for our analytical and numerical results.

III. AN ARCHETYPAL EXAMPLE

Within the general class of potentials (1), a key, well-studied example is the massless $\lambda\phi^4$ model coupled quadratically to a massless χ [17,18]. The conformal potential in this case is

$$\mathcal{V}(\tilde{\phi}, \tilde{\chi}) = \frac{\lambda}{4} \tilde{\phi}^4 + \frac{g^2}{2} \tilde{\phi}^2 \tilde{\chi}^2. \quad (11)$$

We switch to dimensionless time $x \equiv \sqrt{\lambda} \varphi_0 \eta$ where $\varphi_0 \simeq 0.3$ is the value of $\tilde{\phi}$ at the start of preheating corresponding to $x_0 \simeq 2.44$ [17]. In this model, the time-averaged equation of state is that of radiation and the scale factor obeys:

$$a \simeq \left(\frac{2\pi\lambda}{3}\right)^{1/2} \varphi_0^2 \eta = \left(\frac{2\pi}{3}\right)^{1/2} \varphi_0 x. \quad (12)$$

This implies that $a'' \simeq 0$. Define $\tilde{\varphi} \equiv \tilde{\phi}/\varphi_0$. Then $\tilde{\varphi}$ satisfies:

$$\tilde{\varphi}'' + \tilde{\varphi}^3 + \frac{g^2}{\lambda\varphi_0} \tilde{\chi}^2 \tilde{\varphi} = 0, \quad (13)$$

which during the first stage of preheating when $\tilde{\chi} \simeq 0$, has the oscillating solution [5,17] $\tilde{\varphi}(x) = cn(x-x_0, 1/\sqrt{2})$ where cn is the Jacobi elliptic cosine, while

$$\tilde{\chi}'' + \frac{g^2}{\lambda} \tilde{\varphi}^2 \tilde{\chi} = 0. \quad (14)$$

The \tilde{Q}_k^i satisfy:

$$\begin{aligned} \tilde{Q}_k^{\varphi''} + [\kappa^2 + 3\tilde{\varphi}^2] \tilde{Q}_k^{\varphi} &= -2 \frac{g^2}{\lambda\varphi_0} \tilde{\varphi} \tilde{\chi} \tilde{Q}_k^{\chi} \\ &+ \frac{8\pi}{a} \left(\tilde{M}^{\varphi\varphi} \tilde{Q}_k^{\varphi} + \tilde{M}^{\varphi\chi} \tilde{Q}_k^{\chi} \right) \end{aligned} \quad (15)$$

$$\begin{aligned} \tilde{Q}_k^{\chi''} + \left[\kappa^2 + \frac{g^2}{\lambda} \tilde{\varphi}^2 \right] \tilde{Q}_k^{\chi} &= -2 \frac{g^2}{\lambda\varphi_0} \tilde{\varphi} \tilde{\chi} \tilde{Q}_k^{\varphi} \\ &+ \frac{8\pi}{a} \left(\tilde{M}^{\varphi\chi} \tilde{Q}_k^{\varphi} + \tilde{M}^{\chi\chi} \tilde{Q}_k^{\chi} \right), \end{aligned} \quad (16)$$

where $\kappa^2 \equiv k^2/\lambda\varphi_0^2$. These equations split naturally into terms which are expansion-invariant (EI), and those which are non-EI which come from the \tilde{M}^{ij} in eq. (9) and which depend explicitly on a . We now consider these equations with and without χ backreaction.

1. Case 1: Reduction to generalized Lamé form

Let us first study the equations (13–16) without any backreaction^{††}, i.e. using $\tilde{\varphi} = cn(x, 1/\sqrt{2})$. This eliminates all terms on the RHS except that containing $\tilde{M}^{\varphi\varphi}$. We verified numerically that this term provides only a

^{††}By backreaction we mean including terms involving products of the background χ field and a genuine, inhomogeneous, first order variable such as \tilde{Q}_k^{χ} . While $\tilde{\chi}$ is homogeneous, so that the product is ostensibly first order, $\tilde{\chi}$ is vanishingly small during inflation. Hence it is numerically irrelevant at the start of preheating. Once $\tilde{\chi}$ grows large it cannot be neglected and leads to the termination of the resonance (see Fig. 3).

weak, super-Hubble, *growing* mode to \tilde{Q}_k^{φ} . The system is decoupled, exact Floquet theory can be applied to the \tilde{Q}_k^{χ} equation and we are guaranteed solutions growing exponentially as $e^{\mu_{\kappa} x}$ for values of κ and g^2/λ inside resonance bands; see Fig. (1). Upon neglect of $\tilde{M}^{\varphi\varphi}$, Floquet theory can also be applied to the \tilde{Q}_k^{φ} equation.

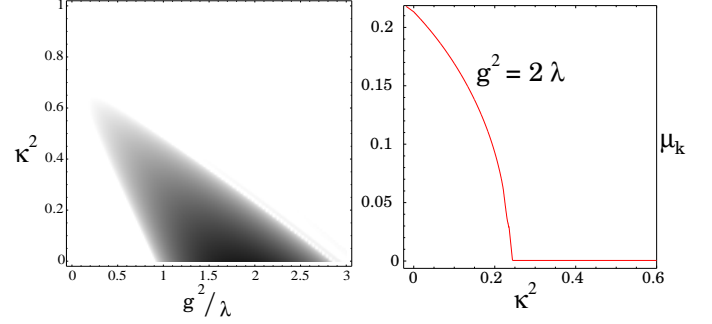


FIG. 1. Floquet index, μ_{κ} , instability chart. White represents $\mu_{\kappa} = 0$, darker greys represent increasing μ_{κ} (left). μ_{κ} vs $\kappa^2 \equiv k^2/\lambda\varphi_0^2$ for $g^2/\lambda = 2$ (right). Both plots are for the generalized Lamé equation given by Eq. (16) with the RHS set to zero and $\tilde{\varphi} = cn(x, 1/\sqrt{2})$.

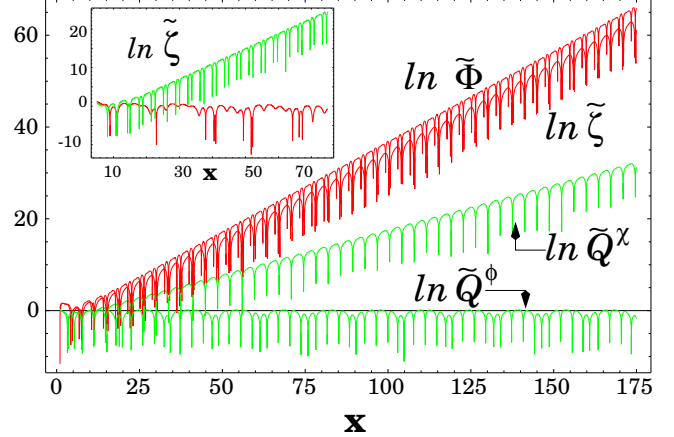


FIG. 2. The gauge-invariant metric perturbations $\tilde{\Phi}$ and \tilde{Q}_k^{χ} , \tilde{Q}_k^{φ} for the super-Hubble mode $\kappa^2 = 10^{-20}$ and $g^2/\lambda = 2$. Note that $\tilde{\Phi}_k$ grows with Floquet index $\mu \simeq 0.357$, much larger than the maximum possible in preheating neglecting metric perturbations which is bounded for all values of g^2/λ to be less than or equal to 0.238. **Inset:** $\tilde{\zeta}_k$ for $g^2/\lambda = 2$ ($\tilde{\zeta}_k$ exponentially growing) and $g^2 = 0$ ($\tilde{\zeta}_k$ constant).

In fact the equations are of generalized Lamé form in *Minkowski* space, as studied in depth by Greene *et al* [17] and Kaiser [18]. The Floquet index μ_{κ} is shown in Fig. (1) as a function of g^2/λ and κ^2 . For $g^2/\lambda = 2$ it reaches its maximum value of $\mu_{\kappa} \simeq 0.238$ at $\kappa^2 = 0$, which coincides with the global maximum of the Floquet index. Further, there exist an infinite number of resonance bands in g^2/λ for which $\mu_{\kappa=0} > 0$, showing that for large regions of parameter space there is essentially exponential growth of super-Hubble metric perturbations.

This provides the first conclusive analytical proof of this fact, confirming previous numerical work [13,14].

In Fig. (2) we plot $\tilde{\Phi}_k$, \tilde{Q}_k^φ and \tilde{Q}_k^χ for $\kappa^2 = 10^{-20}$ and $g^2/\lambda = 2$. Note the perfect exponential growth of \tilde{Q}_k^χ , while \tilde{Q}_k^φ simply oscillates with constant amplitude since it corresponds to $g^2/\lambda = 3$ for which there is only a sub-Hubble resonance band. The Floquet index for \tilde{Q}_k^χ is as expected from the Lamé analysis, while the Floquet index for $\tilde{\Phi}_k$ is much larger, around 0.357. This is expected from the constraint equation (8) which shows that $\tilde{\Phi}'$ is sensitive to the product $\tilde{\chi}\tilde{Q}_k^\chi$, with *both* $\tilde{\chi}$ and \tilde{Q}_k^χ growing exponentially.

Expressed in dimensionless time and neglecting the $a'' \simeq 0$ term in eq. (10), $\tilde{\zeta}$ becomes

$$\tilde{\zeta}_k = \tilde{\Phi}_k + \sqrt{\lambda}\varphi_0 x \tilde{\Phi}'_k, \quad (17)$$

which shows that $\tilde{\zeta}_k$ also grows exponentially, sharing the same Floquet index as $\tilde{\Phi}_k$ before backreaction shuts off the resonance. This is evident in the inset of Fig. (2) where we show $\tilde{\zeta}$ for $g^2/\lambda = 0$ (no growth) and $g^2/\lambda = 2$ (exponential growth).

Finally, note that the effective resonance parameter $q_{eff} = g^2/\lambda$ can be as small as $q_{eff} = 2$ for the Floquet index to reach its global maximum and for super-Hubble preheating to be strong. This means that the χ field is not *necessarily* suppressed during inflation [23]. This is discussed in detail in section (IV). The suppression also does not apply in wide classes of other preheating models [15].

2. Case 2: Backreaction included

We now consider the full equations (13 – 16) without approximation. Fig. (3) shows the results of simulations including backreaction on the inflaton. For $x < 50$, the analysis in *case 1*, presented above, applies. However, when $\ln \tilde{\chi} \sim O(1)$, the backreaction in Eq. (13) becomes important, changing the amplitude and frequency of $\tilde{\varphi}$ oscillations as shown in the inset of Fig. (3). This pushes $\tilde{\chi}$ into a region of narrow resonance with $\mu_{\tilde{\chi}} \simeq 0.05$.

Importantly, however, the metric perturbations do *not* stop growing once χ moves into the region of narrow resonance. The couplings between Eqs. (15) and (16) cause \tilde{Q}_k^φ to begin growing exponentially and *synchronisation* occurs: for $x > 50$ the perturbations $\tilde{\Phi}_k$, \tilde{Q}_k^χ and \tilde{Q}_k^φ all grow with the *same*, large, Floquet index. This shows that any analysis based on the decoupled equations one finds in the absence of metric perturbations must be considered with caution, since it would (1) underestimate the Floquet indices and (2) incorrectly predict no growth of \tilde{Q}_k^φ and $\delta\tilde{\varphi}_k$ for $\kappa^2 \ll 1$. From Eq. (7) one sees that the field fluctuations $\delta\tilde{\varphi}_k$ grow with the same, large Floquet index as the metric perturbations.

Backreaction finally ends the growth of the metric perturbations around $x \sim 135$ when the amplitude of $\tilde{\varphi}$ drops and the inflaton effective mass dramatically increases (see Fig. 3).

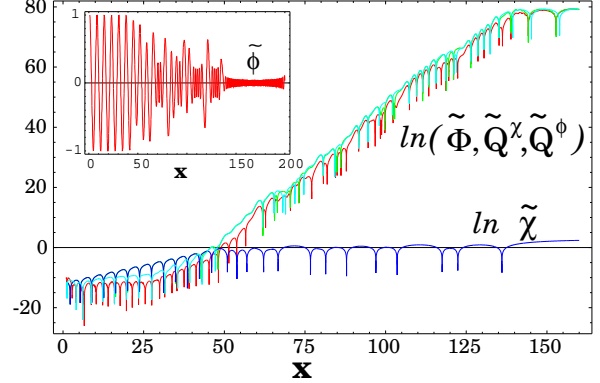


FIG. 3. The field $\tilde{\chi}$ and the gauge-invariant metric perturbations $\tilde{\Phi}_k$, \tilde{Q}_k^χ and \tilde{Q}_k^φ for $\kappa^2 = 10^{-20}$ and $g^2/\lambda = 2$. Note the onset of backreaction in $\tilde{\chi}$ at $x \simeq 50$ and the synchronisation in the metric perturbations for $x \geq 50$. **Inset:** The evolution of the inflaton condensate $\tilde{\varphi}$: note the drop in amplitude and increase in oscillation frequency at $x \simeq 50$ and 135.

IV. PREHEATING INITIAL CONDITIONS

The key point that arises from having resonant growth of the super-Hubble metric modes with $\kappa \ll 1$ for $1 \leq g^2/\lambda < 3$ is its relation to initial conditions at the start of preheating. Convolution of these two will yield the post-preheating power spectrum.

The χ effective mass is given by

$$m_{\chi,eff}^2 = g^2\phi^2. \quad (18)$$

Ignoring the metric perturbations during inflation where they are small, one can solve the Klein-Gordon equation in de Sitter spacetime for the χ modes in terms of Hankel functions [24]:

$$\chi_k = c_1 \sqrt{\eta} H_\nu^{(2)}(k\eta) + c_2 \sqrt{\eta} H_\nu^{(1)}(k\eta), \quad (19)$$

where $c_1 = \sqrt{\pi}/2$, $c_2 = 0$ corresponds to the adiabatic vacuum state and the order, ν , of the Hankel function is the root of

$$\nu^2 = \frac{9}{4} - \frac{m_\chi^2}{H^2} - 12\xi. \quad (20)$$

Here ξ is the non-minimal coupling of the χ field to the curvature, which we have put to zero throughout so far. Asymptotically, as $k\eta \rightarrow 0$, the Hankel functions are [25]

$$H_\nu^{(1,2)}(k\eta) \rightarrow \frac{i}{\pi} \Gamma(\nu) \left(\frac{k\eta}{2} \right)^{-\nu}, \quad (21)$$

so that if $\text{Re}(\nu) > 0$, the mode functions are infra-red divergent. However, when $\text{Re}(\nu) = 0$, it is simple to show that the mode functions are finite as $k\eta \rightarrow 0$ and are essentially k -independent. Further, if $\text{Re}(\nu) = 0$, the modes decay away as $a^{-3/2}$, that is, exponentially in proper time. This is the origin of the claim that preheating has no effect on super-Hubble metric perturbations [23].

Now, in our case, since $\chi = 0$ minimises the potential at fixed ϕ , the χ field has no potential energy during inflation and $H^2 \simeq 2\pi\lambda\phi^4/3$. Hence when $\xi = 0$

$$\nu^2 = \frac{9}{4} - \frac{3g^2}{2\pi\lambda\phi^2}, \quad (22)$$

hence for $g^2/\lambda = 2$, $\nu^2 < 0$ when $\phi < 2/\sqrt{3\pi} \sim 0.7M_{pl}$. Now since preheating starts at around $\phi \sim 0.3M_{pl}$, $\nu^2 > 0$ during most of inflation. This case differs radically from the case $g^2/\lambda \gg 1$, where ν is complex many e-foldings from the end of inflation. Hence, while the suppression of χ modes is very strong when $g^2/\lambda \gg 1$ [23], for $g^2/\lambda \in [1, 3)$ there is little suppression of cosmological χ modes.

Before discussing the post-preheating power spectrum we make two points. Firstly, our potential, Eq. (11), is not meant to be generic, and hence specific results should be taken as proofs of principle rather than as generic. In particular, the suppression of χ modes when $g^2/\lambda \gg 1$ is absent in preheating with more general potentials [15].

Secondly, the suppression that occurs for $g^2/\lambda \gg 1$ and $\chi = 0$ (minimally coupling), does not *necessarily persist* when $\xi \neq 0$. From Eq. (20) one sees that $\nu^2 > 0$ is compatible with $m_\chi^2 \gg H^2$ (and hence little or no suppression) for large negative couplings, ξ , to the curvature.

In fact, for sufficiently large negative ξ , $\nu > 3/2$ and the resulting power spectrum has most power at *large* scales – a red tilt – and field variances receive their dominant contribution from the super-Hubble modes [24]. Large non-minimal couplings in fact lead to their own strong resonances due to the oscillation of the Ricci scalar during preheating [16,26].

V. THE χ POWER SPECTRUM

While our main aim in this paper has been to provide analytical evidence for the existence of super-Hubble resonances, we turn here to the question of the post-preheating power spectrum.

The power spectrum for an arbitrary field X is defined by

$$P_X(k) \equiv \frac{k^3}{2\pi^2} \langle |X_k|^2 \rangle, \quad (23)$$

where $\langle \cdot \rangle$ denotes ensemble averaging (equivalently volume averaging if X is ergodic). From Eqs. (19,21), the

power-spectrum for χ at the end of inflation in the limit $k\eta \rightarrow 0$ is thus given by:

$$P_\chi(k, \eta) = \frac{k^3}{8\pi^3} \eta \Gamma^2(\nu) \left(\frac{k\eta}{2} \right)^{-2\nu}. \quad (24)$$

This clarifies the situation immediately. For ν complex, the power spectrum is steep $\propto k^3$. However, for ν real, the power spectrum is closer to Harrison-Zel'dovich with spectral index

$$n = 3 - 2\nu, \quad (25)$$

which tends to zero as $\nu \rightarrow 3/2$ ($\phi \rightarrow \infty$). Now it is true that ν becomes complex at the end of inflation, but since we are interested in the $k \rightarrow 0$ part of the spectrum, the appropriate value of ν is the one at early stages: $\phi \geq 2M_{pl}$, for which $\nu \geq \sqrt{2}$, leaving only a mild blue tilt.

The power spectrum (24) is then modified by preheating in a rather trivial way: it is multiplied by a factor $\sim e^{2\mu\Delta x}$, where Δx is the time spent in resonance. For $\kappa^2 \ll 1$, modes, this transfer function is almost independent of k since the Floquet index only changes significantly over $\Delta\kappa \sim O(1)$.

The above discussion paves the way for direct evaluation of the impact of preheating on large scales. The obvious question remains: “how much of the g^2/λ phase space will leave an imprint on the CMB?” In this model we may investigate this question analytically without relying on numerical studies.

A. Dividing up the g^2/λ parameter space

For cosmological implications we are typically only interested in the $\kappa^2 \leq 10^{-50}$ section of the $\kappa^2 - g^2/\lambda$ chart, Fig. (1). It is relatively easy to show that the nodes of the resonance bands (where $\mu_{\kappa=0}$ becomes non-zero) are given by $g^2/\lambda = n(n+1)/2$, with n natural. It is then easy to show that asymptotically for large g^2/λ , half of all values of g^2/λ lie in resonance bands, and half lie in stable regions.

However, if $\xi = 0$, only for a much smaller measure of g^2/λ values do the cosmological modes dominate the backreaction integral, immediately ruling out those values of g^2/λ . This clinical division of the g^2/λ parameter space is due to the conformal invariance of our model. For generic models with massive fields the modes will move through the maxima of the resonance bands and the issue of which couplings are ruled out is more subtle, and probably needs to be done numerically.

On the other hand, modes near $g^2/\lambda = 1$ or 3 , will experience super-Hubble growth and are also little suppressed during inflation. However they are unlikely to go nonlinear before backreaction becomes important (since sub-Hubble modes grow the quickest - see Fig 1). Whether these regions of the resonance band are ruled

out is therefore more subtle, but they are expected to leave an imprint on the CMB.

For $g^2/\lambda \gg 1$ ($\xi = 0$), the χ spectrum is suppressed at small k and although super-Hubble modes experience resonant growth, backreaction may well shut off the resonance before they become important and affect the power spectrum in any significant way. Equivalently $n \simeq 3$ in Eq. (25) [23].

The g^2/λ parameter space is thus divided into four sub-regions: (I) regions where backreaction comes from cosmological modes going nonlinear. This is a small set in the conformal model and can be ruled out directly. They form narrow regions around the maxima of the Floquet index at $g^2/\lambda = 2, 8, \dots$ (II) regions with $0 < \mu_{\kappa=0} < \mu_{\text{maximum}}$, i.e. in the wings of the resonance bands. The Q_χ variance is dominated by sub-Hubble modes but they can nevertheless leave an imprint on the CMB if $g^2/\lambda < 10$. (III) Regions with $\mu_{\kappa=0} > 0$ but $g^2/\lambda > 10$ where suppression during inflation is strong and little effect is seen. (IV) Stable bands where $\mu_{\kappa=0} = 0$ and backreaction is irrelevant.

VI. χ SELF-INTERACTION

We now consider the dual issue of the possible effects of the rapidly growing metric perturbations on particle production.

A result from early studies of preheating, believed to be rather robust, was that χ self-interaction lead to a rapid shut-off of the resonance and inefficient decay of the inflaton [22]. This is easy to understand in the absence of metric perturbations since the self-interaction acts like an effective mass $m_{\chi, \text{eff}}^2 \sim 3\lambda_\chi \langle \chi^2 \rangle$, which pushes all modes out of the dominant first resonance band and essentially ends the resonance. This is evident in the inset of Fig. (4) which shows $\tilde{\Phi}$ in a model with $\lambda_\chi \chi^4/4$ self-interaction added to the potential (11), with $\lambda_\chi = 10^{-8}\lambda$ for *case 1*, i.e. the equations (15,16) with all terms on the RHS's set to zero and $\tilde{\varphi} = cn(x, 1/\sqrt{2})$. Even with this tiny self-interaction, backreaction appears to be extremely efficient at ending the resonant growth of super-Hubble fluctuations.

This, however, is an artifact of neglecting metric perturbations which manifests itself as the couplings between the equations (15, 16). The main graphs in Fig. (4) show a very different behaviour, even when a much larger self-interaction, $\lambda_\chi = 10^2\lambda$, is used to emphasise the discrepancies with the old picture. The metric and field fluctuations continue to grow exponentially long after the $\tilde{\chi}$ field has ended its growth. How is this to be understood?

The growth of χ forces it out of the resonance band and causes it to stop growing before it has a chance to affect the inflaton evolution through Eq. (13). This is in complete contrast to the case in Fig. (3). Since it is

the amplitude and frequency of inflaton oscillations that predominantly controls the resonant growth of field and metric fluctuations, the large χ self-interaction actually aids the growth of metric perturbations in the sense that it removes the means of backreaction on the inflaton condensate ^{‡‡}.

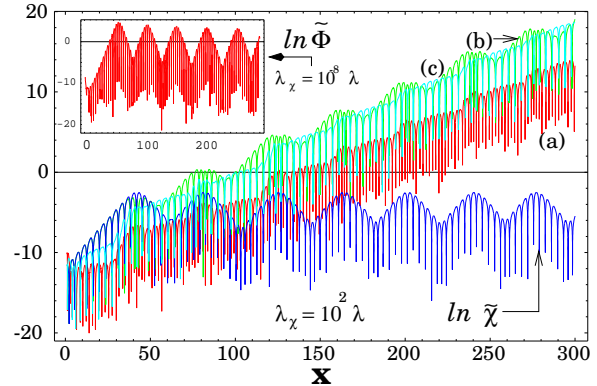


FIG. 4. Evolution of metric perturbations with $\tilde{\chi}$ self-interaction: (a) $\tilde{\Phi}_k$ (b) \tilde{Q}_k^χ (c) \tilde{Q}_k^φ and the $\tilde{\chi}$ field, for $\lambda_\chi = 10^2\lambda$, $\kappa^2 = 10^{-20}$ and $g^2/\lambda = 2$. While self-interaction stops $\tilde{\chi}$ from growing, it fails to stop exponential growth of the metric perturbations. **Inset:** $\tilde{\Phi}_k$ evolution with all terms on the RHS of Eq's (15,16) set to zero, and $\lambda_\chi = 10^{-8}\lambda$. The apparent shut-off of the resonance is an artifact due to neglect of backreaction and coupling between \tilde{Q}_k^φ and \tilde{Q}_k^χ .

This provides an elegant example of both the qualitative and quantitative importance of including metric perturbations in preheating. This is true not only when one is interested in understanding the spectrum of perturbations for the CMB, but also when calculating relic abundances of particles produced during preheating. Relevant examples are the abundances of gravitinos, GUT bosons and super-heavy dark matter whose decays may contribute to the observed ultra-high energy cosmic ray flux [27].

VII. CONCLUSIONS

In conclusion, we have attempted to convince the reader of a number of controversial possibilities: (1) gauge-invariant, super-Hubble scalar metric perturbations can grow exponentially, modulo power-law behaviour, in an *expanding* universe. We did this by reducing the gauge-invariant equations to generalised Lamé form. (2) The quantity ζ , often used to transfer the power

^{‡‡}Using $\langle \tilde{\chi}^2 \rangle$ instead of $\tilde{\chi}^2$ in Eq. (13) will cause the field fluctuations in Fig. (4) to stop growing at some large, but finite, value. However, our main point is to stress the new channels through which the resonance can proceed when metric perturbations are included.

spectrum of fluctuations to decoupling by virtue of its constancy, may be useless in general during preheating due to its exponential amplification. (3) Metric perturbations can significantly alter estimates for the maximum variances of field fluctuations needed for accurate prediction of relic particle abundances.

This last point is particularly transparent when the inflaton decay products exhibit strong self-interaction. In the absence of metric perturbations the self-interaction stops preheating almost completely [22]. When metric perturbations are included however, the resonances proceed through new channels and can be strong even at large self-interaction, greatly enhancing final field variances and number densities. These issues will be discussed in depth in [19].

The issue of the post-preheating χ power spectrum has been discussed as a function of the coupling g^2/λ . We show that while there are an infinite number of super-Hubble resonance bands, only for a small range of values, $g^2/\lambda < 10$, can one expect modifications of standard inflationary predictions for cosmology since these values are partially or completely protected from strong suppression during inflation. In more general, non-conformal, models the situation will be more complex as modes are pulled through resonance bands by the expansion. Crucially the mode suppression mechanism may also be absent [15].

To end we discuss an issue little studied to date. Eliminating all other perturbation variables, the Φ equation of motion typically has singular coefficients $\propto \dot{\phi}^{-1}$ [1,2]. Indeed, this was the main original motivation for using the Sasaki-Mukhanov variables [10,11]. Insight into Φ evolution is provided by the duality between the Klein-Gordon and 1-d spatial Schrödinger equations exploited in [21]. In this approach the resonance band structure in k -space becomes dual to the spectrum of the corresponding Schrödinger equation. Now consider the dimensionless Scarf Hamiltonian:

$$H\psi \equiv \left(-\frac{d^2}{dz^2} + \frac{A}{\sin^2 z} \right) \psi(z) = \lambda\psi \quad (26)$$

whose potential is singular periodically when $\sin z = 0$. This Hamiltonian is known to have a spectrum with band structure for $-1/4 < A < 0$ and discrete eigenvalues otherwise [28]. On the complement of the spectrum, $\mathbf{R} - \sigma(H)$, the corresponding Floquet index is real and positive [21]. This explains why the behaviour of $\tilde{\Phi}_k$ can be qualitatively similar to that of \tilde{Q}_k^x and \tilde{Q}_k^y , which obey *non-singular* equations.

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